Section 5.7

The Substitution Method

by Joseph Phillip Brennan
Jila Niknejad

The Substitution Method

Example 1: Evaluate $\int 3e^{3x}\,dx$.

The idea is to rewrite the integral in a simpler form by changing variables.

Set $u = 3x$, so that $e^{3x} = e^u$.
But what do we do with the dx in the integral?

$$u = 3x \quad \Rightarrow \quad \frac{du}{dx} = 3 \quad \Rightarrow \quad du = 3\,dx \quad \Rightarrow \quad dx = \frac{1}{3}\,du$$

Using the highlighted substitutions, we get

$$\int 3e^{3x}\,dx \;=\; \int e^u\,du \;=\; e^u + C \;=\; \boxed{e^{3x} + C.}$$

The Substitution Method

Example 2: Evaluate $\int z \sin(z^2)\, dz$.

The Substitution Method

The Substitution Method for indefinite integrals $\int f(x)\,dx$:

(1) Find an appropriate expression $g(x)$ that appears as a part of the integrand.

(2) Let $u = g(x)$ and $du = g'(x)\,dx$.

(3) Rewrite the integral in terms of u.

Hopefully the new expression will be easier to integrate!

(4) When you have finished integrating in terms of u, substitute $u = g(x)$ back in to write the final answer in terms of x.

Warnings:

(1) Always include the dx or du when you write an integral. This is necessary to keep track of where you are in the process.

(2) Never mix two variables in the same integral!

The Substitution Method and the Chain Rule

In Example 1, we calculated

$$\int 3e^{3x}\,dx = e^{3x} + C.$$

To confirm, the Chain Rule says that

$$\frac{d}{dx}\left(e^{3x} + C\right) = 3e^{3x}.$$

In Example 2, we calculated

$$\int z\sin(z^2)\,dz = -\frac{1}{2}\cos(z^2) + C.$$

To confirm, the Chain Rule says that

$$\frac{d}{dz}\left(-\frac{1}{2}\cos(z^2) + C\right) = z\sin(z^2).$$

The Substitution Method: More Examples

Example 3: Evaluate $\int 4(x+1)(x^2+2x)^{2/3}\,dx$.

The Substitution Method: More Examples

Example 4: $\int (y+5)\sqrt{10y+y^2}\,dy$

The Substitution Method: More Examples

Example 5: $\displaystyle\int \frac{x}{\sqrt{x^2+9}}\,dx$

The Substitution Method: More Examples

Example 6: $\int (t+2)(t+1)^{1/4}\,dt$ $\quad u = t+1 \quad du = dt$

The Substitution Method: More Examples

Example 7: $\int x^3 \sqrt{x^2+4}\, dx$ $\quad u = x^2 + 4 \quad du = 2x\, dx$

The Substitution Method: More Examples

Example 8: $\displaystyle\int \frac{dx}{x\ln|x|}$

Example 9: $\displaystyle\int \tan(x)\,dx$

The Substitution Method for Definite Integrals

Back in Example 1, we evaluated the **indefinite** integral

$$\int 3e^{3x}\,dx$$

by substituting $u = 3x$, $du = 3\,dx$ to get

$$\int 3e^{3x}\,dx \;=\; \int e^u\,du \;=\; e^u + C \;=\; e^{3x} + C.$$

We can evaluate **definite** integrals either by converting back to the original variable x ...

$$\int_1^2 3e^{3x}\,dx \;=\; e^{3x}\Big|_1^2 \;=\; e^6 - e^3$$

... or by changing the limits of integration from x-values to u-values:

$$\int_1^2 3e^{3x}\,dx \;=\; \int_{u(1)}^{u(2)} e^u\,du \;=\; e^u\Big|_3^6 \;=\; e^6 - e^3.$$

The Substitution Method for Definite Integrals

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Example 10: $\int_{-1}^{4} 2(5-3x)^2\, dx$

Example 11: Evaluate $\int_0^1 \sqrt[3]{1+7x}\,dx$.

Example 12: Evaluate $\int_0^2 \frac{x^3}{x^2+1}\,dx$.

Example 13: If f is continuous and $\int_0^4 f(x)\,dx = 2$, find $\int_0^2 f(2x)\,dx$.

Substitution and Indefinite Integrals

Figure shows R_4 for the integral $\int_0^2 e^{3x}\,dx$ and the integral resulting from it by the substitution $u = 3x$.

$$\int_0^2 e^{3x}\,dx = \int_0^6 \frac{1}{3} e^u\,du$$